Rewarding Learning

ADVANCED
General Certificate of Education
2024

Further Mathematics

Assessment Unit A2 1
assessing
Pure Mathematics

[AFM11]

WEDNESDAY 22 MAY, AFTERNOON

MARK SCHEME

GCE ADVANCED/ADVANCED SUBSIDIARY (AS) FURTHER MATHEMATICS

Introduction

The mark scheme normally provides the most popular solution to each question. Other solutions given by candidates are evaluated and credit given as appropriate; these alternative methods are not usually illustrated in the published mark scheme.

The marks awarded for each question are shown in the right-hand column and they are prefixed by the letters **M**, **W** and **MW** as appropriate. The key to the mark scheme is given below:

M indicates marks for correct method.

W indicates marks for working.

MW indicates marks for combined method and working.

The solution to a question gains marks for correct method and marks for an accurate working based on this method. Where the method is not correct no marks can be given.

A later part of a question may require a candidate to use an answer obtained from an earlier part of the same question. A candidate who gets the wrong answer to the earlier part and goes on to the later part is naturally unaware that the wrong data is being used and is actually undertaking the solution of a parallel problem from the point at which the error occurred. If such a candidate continues to apply correct method, then the candidate's individual working must be followed through from the error. If no further errors are made, then the candidate is penalised only for the initial error. Solutions containing two or more working or transcription errors are treated in the same way. This process is usually referred to as "follow-through marking" and allows a candidate to gain credit for that part of a solution which follows a working or transcription error.

Positive marking:

It is our intention to reward candidates for any demonstration of relevant knowledge, skills or understanding. For this reason we adopt a policy of **following through** their answers, that is, having penalised a candidate for an error, we mark the succeeding parts of the question using the candidate's value or answers and award marks accordingly.

Some common examples of this occur in the following cases:

(a) a numerical error in one entry in a table of values might lead to several answers being incorrect, but these might not be essentially separate errors;

(b) readings taken from candidates' inaccurate graphs may not agree with the answers expected but might be consistent with the graphs drawn.

When the candidate misreads a question in such a way as to make the question easier only a proportion of the marks will be available (based on the professional judgement of the examining team).

			AVAILABLE MARKS
1	Test $n = 1$		
	$\mathbf{A}^1 = \begin{pmatrix} 3^1 & 0 \\ \dfrac{(3^1 - 1)a}{2} & 1 \end{pmatrix}$	M1	
	$= \begin{pmatrix} 3 & 0 \\ a & 1 \end{pmatrix}$	W1	
	Therefore true for $n = 1$		
	Assume true for $n = k$	M1	
	i.e. $\mathbf{A}^k = \begin{pmatrix} 3^k & 0 \\ \dfrac{(3^k - 1)a}{2} & 1 \end{pmatrix}$		
	$\Rightarrow \mathbf{A}^{k+1} = \begin{pmatrix} 3 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ \dfrac{(3^k - 1)a}{2} & 1 \end{pmatrix}$	M1 W1	
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3^k a + \dfrac{(3^k - 1)a}{2} & 1 \end{pmatrix}$		
	$= \begin{pmatrix} 3^{k+1} & 0 \\ \dfrac{2 \times 3^k a + 3^k a - a}{2} & 1 \end{pmatrix}$		
	$= \begin{pmatrix} 3^{k+1} & 0 \\ \dfrac{(3^{k+1} - 1)a}{2} & 1 \end{pmatrix}$	W1	
	Therefore true for $n = k + 1$ Since true for $n = 1$, then by induction true for all $n \geqslant 1$	MW1	7

				AVAILABLE MARKS
2	**(i)**	$\dfrac{1}{(4r-3)(4r+1)} \equiv \dfrac{A}{(4r-3)} + \dfrac{B}{(4r+1)}$	M1 W1	
		$\Rightarrow A(4r+1) + B(4r-3) \equiv 1$	M1	
		$r = -\dfrac{1}{4} \Rightarrow -4B = 1$		
		$B = -\dfrac{1}{4}$	MW1	
		$r = \dfrac{3}{4} \Rightarrow 4A = 1$		
		$A = \dfrac{1}{4}$	MW1	
		$\Rightarrow \displaystyle\sum_{r=1}^{n} \frac{1}{(4r-3)(4r+1)} \equiv \sum_{r=1}^{n} \frac{1}{4}\left(\frac{1}{(4r-3)} - \frac{1}{(4r+1)}\right)$	M1	
		$= \dfrac{1}{4}\left(\dfrac{1}{1} - \dfrac{1}{5}\right)$ $+ \dfrac{1}{4}\left(\dfrac{1}{5} - \dfrac{1}{9}\right)$ $+ \dfrac{1}{4}\left(\dfrac{1}{9} - \dfrac{1}{13}\right)$ $+ \cdots$ $\cdots$	MW1	
		$+ \dfrac{1}{4}\left(\dfrac{1}{4n-7} - \dfrac{1}{4n-3}\right)$ $+ \dfrac{1}{4}\left(\dfrac{1}{4n-3} - \dfrac{1}{4n+1}\right)$	MW1	
		$= \dfrac{1}{4}\left(1 - \dfrac{1}{4n+1}\right)$	W1	
	(ii)	$\displaystyle\sum_{r=1}^{\infty} \frac{1}{(4r-3)(4r+1)} = \lim_{n\to\infty}\left[\frac{1}{4}\left(1 - \frac{1}{4n+1}\right)\right]$	M1	
		$= \dfrac{1}{4}(1-0)$		
		$= \dfrac{1}{4}$	MW1	11

			AVAILABLE MARKS
3 **(i)**	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$	MW1	
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$	MW1	
	$\Rightarrow e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$	M1	
	$= x - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \frac{x^5}{24}$	W1	
	$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$	W1	
(ii)	$\cosh x \sin x = \frac{1}{2}(e^x + e^{-x}) \sin x$	M1	
	$e^{-x} \sin x = -e^{-x} \sin(-x)$	M1	
	$= -\left(-x + x^2 - \frac{x^3}{3} + \frac{x^5}{30}\right)$		
	$= x - x^2 + \frac{x^3}{3} - \frac{x^5}{30}$	W1	
	$\Rightarrow \cosh x \sin x = \frac{1}{2}\left(x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + x - x^2 + \frac{x^3}{3} - \frac{x^5}{30}\right)$	M1	
	$= x + \frac{x^3}{3} - \frac{x^5}{30}$	W1	10
4	$\int_{\sqrt{2}}^{\sqrt{5}} \cosh^{-1} x \, dx$		
	$u = \cosh^{-1} x \qquad \frac{dv}{dx} = 1$	M1 W1	
	$\frac{du}{dx} = \frac{1}{\sqrt{x^2 - 1}} \qquad v = x$	W2	
	$\int_{\sqrt{2}}^{\sqrt{5}} \cosh^{-1} x \, dx = [x \cosh^{-1} x]_{\sqrt{2}}^{\sqrt{5}} - \int_{\sqrt{2}}^{\sqrt{5}} \frac{x}{\sqrt{x^2 - 1}} dx$	M1 W1	
	$= \left[x \cosh^{-1} x - \sqrt{x^2 - 1}\right]_{\sqrt{2}}^{\sqrt{5}}$	MW1	
	$= \left[\sqrt{5} \cosh^{-1}(\sqrt{5}) - \sqrt{5 - 1}\right] - \left[\sqrt{2} \cosh^{-1}(\sqrt{2}) - \sqrt{2 - 1}\right]$	M1	
	$= \sqrt{5} \cosh^{-1}(\sqrt{5}) - \sqrt{2} \cosh^{-1}(\sqrt{2}) - 1$	W1	9

			AVAILABLE MARKS
5	**(i)** $\frac{dx}{dt} = k\left(\frac{x}{t} - t\right)$	MW1	
	$t = 1, x = 2, \frac{dx}{dt} = 2$		
	$\Rightarrow 2 = k\left(\frac{2}{1} - 1\right)$	M1	
	$k = 2$	W1	
	$\Rightarrow \frac{dx}{dt} = 2\left(\frac{x}{t} - t\right)$		
	$\frac{dx}{dt} - \frac{2x}{t} = -2t$	W1	
	(ii) IF $= e^{-\int \frac{2}{t} dt}$	M1	
	$= e^{-2 \ln t}$		
	$= \frac{1}{t^2}$	W1	
	$\Rightarrow \frac{1}{t^2} \times \frac{dx}{dt} - \frac{1}{t^2} \times \frac{2x}{t} = \frac{1}{t^2} \times -2t$	MW1	
	$\Rightarrow \frac{1}{t^2}\frac{dx}{dt} - \frac{2x}{t^3} = -\frac{2}{t}$		
	Integrate to give		
	$\frac{x}{t^2} = \int -\frac{2}{t} dt$	M1	
	$\Rightarrow \frac{x}{t^2} = -2 \ln t + c$	W2	
	$t = 1, x = 2 \Rightarrow 2 = 0 + c$	M1	
	$c = 2$		
	$x = -2t^2 \ln t + 2t^2$		
	$x = 2t^2 (1 - \ln t)$	W1	
	(iii) x is undefined at time $t = 0$ and therefore the initial distance of the particle cannot be found.	MW1	13

6 **(i)** RHS $\equiv \dfrac{\dfrac{2\sinh x}{\cosh x}}{1+\dfrac{\sinh^2 x}{\cosh^2 x}}$ M1 W1

$\equiv \dfrac{2\sinh x\cosh x}{\cosh^2 x+\sinh^2 x}$ MW1

$\equiv \dfrac{2\times\frac{1}{2}(e^x-e^{-x})\times\frac{1}{2}(e^x+e^{-x})}{\frac{1}{4}(e^x+e^{-x})^2+\frac{1}{4}(e^x-e^{-x})^2}$ M1 W1

$\equiv \dfrac{\frac{1}{2}(e^{2x}-e^{-2x})}{\frac{1}{4}(e^{2x}+2+e^{-2x}+e^{2x}-2+e^{-2x})}$

$\equiv \dfrac{\frac{1}{2}(e^{2x}-e^{-2x})}{\frac{1}{2}(e^{2x}+e^{-2x})}$ MW1

$\equiv \dfrac{\sinh 2x}{\cosh 2x}$ W1

$\equiv \tanh 2x$ W1

(ii) $\dfrac{2}{3}=\dfrac{2\tanh y}{1+\tanh^2 y}$ M1

$\Rightarrow 2+2\tanh^2 y=6\tanh y$ W1

$\tanh^2 y-3\tanh y+1=0$ W1

$\tanh y=\dfrac{3\pm\sqrt{5}}{2}$ W1

But $-1<\tanh y<1$

$\Rightarrow \tanh y=\dfrac{3-\sqrt{5}}{2}$ MW1

AVAILABLE MARKS: 13

			Marks	AVAILABLE MARKS
7	**(a)**	AQE $m^2 - pm + (p + 3) = 0$	M1	
		AQE must have equal roots		
		$\Rightarrow p^2 - 4(p + 3) = 0$	M1 W1	
		$p^2 - 4p - 12 = 0$		
		$(p - 6)(p + 2) = 0$		
		$p = -2, \quad p = 6$	W1	
	(b)	AQE $m^2 + 9 = 0$	MW1	
		$\Rightarrow m = \pm 3\mathrm{i}$	W1	
		CF $y = A \cos 3x + B \sin 3x$	M1 W1	
		PI $y = ax^2 + bx + c$	M1	
		$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$	W1	
		$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a$	W1	
		$\Rightarrow 2a + 9(ax^2 + bx + c) \equiv 18x^2 - 5$	M1	
		Compare coefficients	M1	
		$x^2 : \; 9a = 18$		
		$a = 2$	W1	
		$x : \; 9b = 0$		
		$b = 0$		
		Constant $2a + 9c = -5$		
		$c = -1$	W1	
		$\Rightarrow y = A \cos 3x + B \sin 3x + 2x^2 - 1$	MW1	16

		AVAILABLE MARKS
8 **(i)** $I_n = \int x^n \sinh x \, dx$		
$u = x^n \qquad \frac{dv}{dx} = \sinh x$	M1 W1	
$\frac{du}{dx} = nx^{n-1} \quad v = \cosh x$	W2	
$I_n = x^n \cosh x - \int nx^{n-1} \cosh x \, dx$	MW1	
$u = nx^{n-1} \qquad \frac{dv}{dx} = \cosh x$	M1	
$\frac{du}{dx} = n(n-1)\, x^{n-2} \quad v = \sinh x$	W2	
$I_n = x^n \cosh x - \left[nx^{n-1} \sinh x - \int n(n-1)x^{n-2} \sinh x \, dx\right]$	M1	
$I_n = x^n \cosh x - nx^{n-1} \sinh x + n(n-1)I_{n-2}$	W1	
(ii) Area $= \int_0^1 x^4 \sinh x \, dx$	MW1	
Let $I_4 = \int_0^1 x^4 \sinh x \, dx$	M1	
$= 1 \cosh 1 - 4 \sinh 1 + 4(3)I_2$	MW1	
$I_2 = \cosh 1 - 2 \sinh 1 + 2(1)I_0$	MW1	
$I_0 = \int_0^1 \sinh x \, dx$		
$= [\cosh x]_0^1$		
$= \cosh 1 - 1$	MW1	
$\Rightarrow I_2 = \cosh 1 - 2 \sinh 1 + 2 \cosh 1 - 2$		
$= 3 \cosh 1 - 2 \sinh 1 - 2$		
$\Rightarrow I_4 = \cosh 1 - 4 \sinh 1 + 36 \cosh 1 - 24 \sinh 1 - 24$	MW1	
$\Rightarrow I_4 = 37\cosh 1 - 28 \sinh 1 - 24$	MW1	17

			AVAILABLE MARKS
9	**(i)** $x = r\cos\theta \qquad y = r\sin\theta$		
	$r(2\cos\theta + \sin\theta) = 2$		
	$2r\cos\theta + r\sin\theta = 2$	M1 W1	
	$2x + y = 2$	W1	
	Since this is of the form $y = mx + c$ then it represents points on a straight line.	MW1	
	(ii) Line l : P(1, 0) $\quad 1 = \dfrac{2}{2\cos 0 + \sin 0} \quad$ True	M1	
	$Q\left(2, \frac{\pi}{2}\right) \quad 2 = \dfrac{2}{2\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)} \quad$ True	W1	
	Curve C: P(1, 0) $\quad 1 = 1 + \sin 0 \quad$ True		
	$Q\left(2, \frac{\pi}{2}\right) \quad 2 = 1 + \sin\left(\frac{\pi}{2}\right) \quad$ True	MW1	
	Hence P and Q are points of intersection of the curve and the line.		
	(iii) Area of S $= \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2\,d\theta - \text{Area } \Delta OPQ$		
	$\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \sin\theta)^2\,d\theta$	M1 W2	
	$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + 2\sin\theta + \sin^2\theta)\,d\theta$	W1	
	$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right)d\theta$	MW1	
	$\displaystyle = \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$	MW1	
	$\displaystyle = \frac{1}{2}\left[\frac{3\pi}{4} - 0 - 0\right] - \frac{1}{2}[0 - 2 - 0]$		
	$\displaystyle = \frac{3\pi}{8} + 1$	W1	
	Area $\Delta OPQ = \frac{1}{2} \times 1 \times 2$	M1	
	$= 1$	W1	
	Area of S $= \frac{3\pi}{8} + 1 - 1$		
	$= \frac{3\pi}{8}$ square units	MW1	17

10 (a) (i)	$\frac{1}{z} = (\cos\theta + \text{i}\sin\theta)^{-1}$	M1	**AVAILABLE MARKS**
	$= \cos\theta - \text{i}\sin\theta$	W1	
	$z - \frac{1}{z} = \cos\theta + \text{i}\sin\theta - (\cos\theta - \text{i}\sin\theta)$		
	$= 2\text{i}\sin\theta$	MW1	
(ii)	$(2\text{i}\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$	M1	
	$\Rightarrow 2^5\,(\text{i})^5\sin^5\theta$		
	$= z^5 - 5(z^4)\left(\frac{1}{z}\right) + 10(z^3)\left(\frac{1}{z}\right)^2 - 10(z^2)\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 - \left(\frac{1}{z}\right)^5$	M1 W1	
	$32\text{i}\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$	M1 W1	
	$= 2\text{i}\sin 5\theta - 10\text{i}\sin 3\theta + 20\text{i}\sin\theta$	M1	
	$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$	W1	
(b) (i)	$1 + \sqrt{3}\,\text{i} \equiv r(\cos\theta + \text{i}\sin\theta)$	M1	
	$\Rightarrow r\cos\theta = 1$ and $r\sin\theta = \sqrt{3}$		
	$r = 2 \quad \theta = \frac{\pi}{3}$	W2	
	$1 + \sqrt{3}\,\text{i} = 2\left(\cos\left(\frac{\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{3}\right)\right)$		
	$1 - \sqrt{3}\,\text{i} = 2\left(\cos\left(-\frac{\pi}{3}\right) + \text{i}\sin\left(-\frac{\pi}{3}\right)\right)$	MW1	
	$(1 + \sqrt{3}\,\text{i})^n + (1 - \sqrt{3}\,\text{i})^n$		
	$\equiv 2^n\left(\cos\left(\frac{\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{3}\right)\right)^n + 2^n\left(\cos\left(\frac{-\pi}{3}\right) + \text{i}\sin\left(\frac{-\pi}{3}\right)\right)^n$		
	$\equiv 2^n\left(\cos\left(\frac{n\pi}{3}\right) + \text{i}\sin\left(\frac{n\pi}{3}\right)\right) + 2^n\left(\cos\left(\frac{-n\pi}{3}\right) + \text{i}\sin\left(\frac{-n\pi}{3}\right)\right)$	M1 W1	
	$\equiv 2^n\left(2\cos\left(\frac{n\pi}{3}\right)\right)$		
	$\equiv 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$	W1	
(ii)	$(1 + \sqrt{3}\,\text{i})^6 + (1 - \sqrt{3}\,\text{i})^6 = 2^7\cos 2\pi$	M1	
	$= 128$	W1	19

AVAILABLE MARKS

11 **(i)** $\sin^{-1}\left(\frac{3+5\cos x}{5+3\cos x}\right)$ exists provided

$-1 \leqslant \frac{3 + 5\cos x}{5 + 3\cos x} \leqslant 1$ and $5 + 3\cos x \neq 0$ M1

Assume $5 + 3\cos x = 0$ M1

$$\cos x = -\frac{5}{3}$$

which is not feasible

$\Rightarrow 5 + 3\cos x \neq 0$ for all values of x W1

Assume $\frac{3+5\cos x}{5+3\cos x} > 1$ M1

But $5 + 3\cos x$ is always positive

$\Rightarrow$ $3 + 5\cos x > 5 + 3\cos x$

$2\cos x > 2$

$\cos x > 1$

which is not feasible W1

Assume $\frac{3+5\cos x}{5+3\cos x} < -1$

$3 + 5\cos x < -5 - 3\cos x$

$8\cos x < -8$

$\cos x < -1$

which is not feasible MW1

Hence

$\sin^{-1}\left(\frac{3+5\cos x}{5+3\cos x}\right)$ exists for all values of x

(ii) $y = \sin^{-1}\left(\frac{3+5\cos x}{5+3\cos x}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{3+5\cos x}{5+3\cos x}\right)^2}} \times \frac{(5+3\cos x)(-5\sin x)-(3+5\cos x)(-3\sin x)}{(5+3\cos x)^2}$$ M2 W1 M1 W1

$$= \frac{-25\sin x + 9\sin x}{(5+3\cos x)\sqrt{(5+3\cos x)^2-(3+5\cos x)^2}}$$ W1

$$= \frac{-16\sin x}{(5+3\cos x)\sqrt{16-16\cos^2 x}}$$

But $\sin x > 0$ for $0 < x < \pi$

$$\Rightarrow \frac{dy}{dx} = \frac{-16\sin x}{(5+3\cos x)(4\sin x)}$$ MW1

$$= \frac{-4}{5+3\cos x}$$ W1

		AVAILABLE MARKS
(iii) $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{4}{5+3\cos x}\,\mathrm{d}x = -\left[\sin^{-1}\left(\frac{3+5\cos x}{5+3\cos x}\right)\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$	M1 W1	
$= -\sin^{-1}\left(\dfrac{3+5\cos\left(\frac{2\pi}{3}\right)}{5+3\cos\left(\frac{2\pi}{3}\right)}\right) + \sin^{-1}\left(\dfrac{3+5\cos\left(\frac{\pi}{2}\right)}{5+3\cos\left(\frac{\pi}{2}\right)}\right)$	M1	
$= -\sin^{-1}\left(\dfrac{3-\frac{5}{2}}{5-\frac{3}{2}}\right) + \sin^{-1}\left(\frac{3+0}{5+0}\right)$		
$= -\sin^{-1}\left(\frac{1}{7}\right) + \sin^{-1}\left(\frac{3}{5}\right)$	W1	
		18
	Total	**150**